Lecture 4

6.4* - General Exponentials and Logarithms

Let $a>0$, then for any real number x:

$$a^x = e^{x \ln a}$$

Notice that we must have $a>0$ because for example, if $a=-1$ and $x=\frac{1}{2}$, then $a^x = \sqrt{-1}$ which is not defined (not real).

Algebraic Properties

$$a^{x+y} = a^x a^y, \quad a^{x-y} = \frac{a^x}{a^y}, \quad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

e.g.

$$a^{x+y} = e^{(x+y)\ln a} = e^{x\ln a + y\ln a} = e^{x\ln a} e^{y \ln a} = a^x a^y$$

Ex: Simplify

$$\frac{(a^x)^2 a^{x^2+1}}{a^2} = \frac{a^{2x} a^{x^2+1}}{a^2} = a^{2x+x^2+1-2} = \boxed{a^{x^2+2x+1}}$$

$$\left(= a^{(x+1)^2} \right)$$

Differentiation

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = (\ln a)e^{x\ln a} = a^x \ln a$$

so, by the chain rule

$$\frac{d}{dx}(a^{g(x)}) = a^{g(x)}(\ln a)g'(x)$$

Ex: Differentiate $k(x) = x^5 + 5^x$

$$\boxed{k'(x) = 5x^4 + 5^x \ln 5}$$

Integration looks similar to before as well

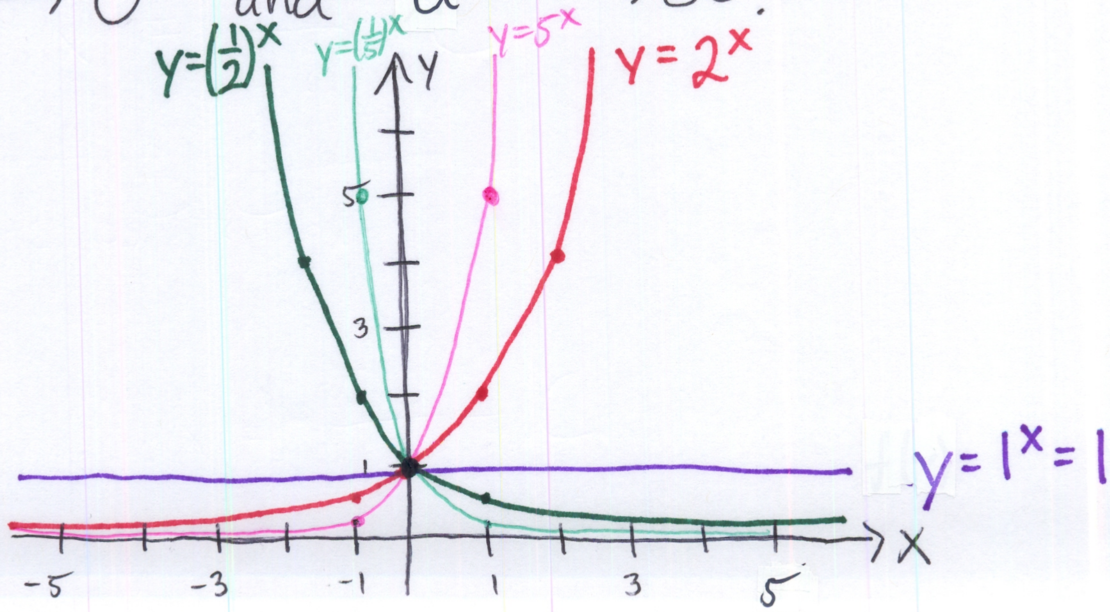
$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

and, by u-substitution

$$\int g'(x) a^{g(x)} \, dx = \frac{a^{g(x)}}{\ln a} + C$$

How does changing the value of a change the graph of $f(x) = a^x$?

- When $a=1$, $f(x) = 1^x = 1$, a horizontal line.
- For $a>1$, a^2, for example, increases as a increases. So, the graph becomes steeper. Also,
  $a^x \xrightarrow{x\to\infty} \infty$ and $a^x \xrightarrow{x\to -\infty} 0$.
- In the case $a<1$, let's write $b = \frac{1}{a}$. Then $b>1$ and $f(x) = a^x = \left(\frac{1}{b}\right)^x = b^{-x}$. So, this is like the previous case, but reflected about the y-axis. So as b increases, meaning that a decreases, the curve gets steeper. Finally
  $a^x \xrightarrow{x\to\infty} 0$ and $a^x \xrightarrow{x\to -\infty} \infty$.



The most general kind of exponential function is of the form $f(x)^{g(x)}$

We can differentiate this in two ways:

- logarithmic differentiation
- writing $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$

Ex: Differentiate $y = (\sin x)^{\ln x}$

$$y = (\sin x)^{\ln x} = e^{(\ln x)\ln(\sin x)}$$

$$y' = e^{(\ln x)\ln(\sin x)} \cdot \frac{d}{dx}\left((\ln x)\ln(\sin x)\right)$$

$$= \boxed{(\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \frac{(\ln x)\cos x}{\sin x}\right)}$$

Ex: Find the limit $\lim_{x\to 0^+} x^{-\ln x}$

$$x^{-\ln x} = e^{\ln(x^{-\ln x})} = e^{-(\ln x)^2}$$

$$\lim_{x\to 0^+} x^{-\ln x} = \lim_{x\to 0^+} e^{-(\ln x)^2} = 0 \quad \text{since}$$

$$\lim_{x\to 0^+} -(\ln x)^2 = -\infty, \text{ since } \lim_{x\to 0^+} \ln x = -\infty$$

As long as $a \neq 1$, the function $f(x) = a^x$ is one-to-one, and so has an inverse, which we call

$$f^{-1}(x) = \log_a x$$

(for example, $\log_e x = \ln x$)

We can interpret $\log_a x$ in terms of $\ln x$ as:

$$y = \log_a x \iff a^y = x \iff y \ln a = \ln a^y = \ln x \iff y = \frac{\ln x}{\ln a}$$

This is the <u>change of base formula</u>:

$$\log_a x = \frac{\ln x}{\ln a}$$

The usual rules for logarithms hold too:

$$\log_a 1 = 0 \qquad \log_a (xy) = \log_a x + \log_a y \qquad \log_a (x^r) = r \log_a x$$

From the change of base formula, we have:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}$$

and the chain rule gives

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{g'(x)}{g(x) \ln a}$$

Ex: Compute

$$\frac{d}{dx}\left(\log_5(xe^x)\right) = \boxed{\frac{e^x + xe^x}{xe^x \ln 5}} \qquad (g(x) = xe^x)$$

Let's make the number e more of a concrete thing:
Take the derivative of $f(x) = \ln x$ at $x = 1$:

$$1 = \frac{1}{1} = f'(1) = \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x\to 0} \ln\left[(1+x)^{1/x}\right]$$

Since e^x is a continuous function, we apply it to both sides and get:

$$e = e^1 = e^{\lim_{x\to 0} \ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0} (1+x)^{1/x}$$

Using this limit, we can approximate e. First, notice that, with the substitution $n = \frac{1}{x}$:

$$e = \lim_{x \to 0} (1+x)^{1/x} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

So, plugging in larger values for n gives a better approximation of e:

$n =$	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.7181459...

The known value of e:

$e = 2.71828\ldots$

(e is an irrational number, in fact, it's what is called transcendental since it is not a zero of a polynomial with rational coefficients.)

Ex: Compute $\lim_{x \to 0} \left(1 + \frac{x}{3}\right)^{1/x}$

Let $m = \frac{x}{3}$, then $x = 3m \Rightarrow \frac{1}{x} = \frac{1}{3m}$

$$\lim_{x \to 0} \left(1 + \frac{x}{3}\right)^{1/x} = \lim_{m \to 0} (1+m)^{1/3m} = \left(\lim_{m \to 0} (1+m)^{1/m}\right)^{1/3}$$

$$= \boxed{e^{1/3}}$$